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Determinants and periodic solutions of delay equations

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Abstract

Finite difference equations may be thought of as discrete analogues of delay equations. Taking this point of view, we give an elementary account of an algebraic determinant identity, due to Burghelea, Friedlander and Kappeler, which relates the determinant of a periodic difference operator to the monodromy of a fundamental solution. The result is applied to a simple class of functional integral operators.

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1. Introduction

In [1] Atiyah related the ζ-function determinant of a Dirac operator on the circle to the holonomy of the orthogonal connection. Burghelea et al. in [3] generalized Atiyah's result to higher order differential operators, (see also [2,4,6–9]), and gave a parallel treatment of the analogous, but entirely algebraic and much more elementary, discrete theory for finite difference operators. Such finite difference operators may

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also be regarded as discrete analogues of functional integral/differential operators, and we shall adopt this point of view in this paper, which is primarily concerned with the algebraic determinant identity of Burghelea, Friedlander and Kappeler. To indicate the elementary nature of this identity, we begin by stating two special cases. The first is well known; special cases of the second occur, for example, in the theory of periodic continued fractions. In the statement K denotes an arbitrary field and $k \geqslant 1$ is a fixed natural number.

Example 1.1. Let $A_j \in M_n(K)$, $1 \leqslant j \leqslant k$, $k > 1$. Then

$$\det\left\{ I_{kn} - \begin{bmatrix} 0 & \cdots & 0 & A_1 \\ A_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdots & 0 & A_k & 0 \end{bmatrix} \right\} = \det\{I_k - A_k A_{k-1} \cdots A_2 A_1\}.$$

Example 1.2. Let $A_j,\ B_j \in\ M_n(K)$, $1 \leqslant j \leqslant k$, $k > 2$. Then

$$\det\left\{ I_{kn} - \begin{bmatrix} 0 & \cdots & 0 & 0 & B_1 & A_1 \\ A_2 & 0 & \cdots & 0 & 0 & B_2 \\ B_3 & A_3 & 0 & \cdots & 0 & 0 \\ 0 & B_4 & A_4 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & B_k & A_k & 0 \end{bmatrix} \right\}$$
$$= \det\left\{ I_{2k} - \begin{bmatrix} 0 & 1 \\ B_k & A_k \end{bmatrix} \begin{bmatrix} 0 & 1 \\ B_{k-1} & A_{k-1} \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ B_2 & A_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ B_1 & A_1 \end{bmatrix} \right\}.$$

Section 2 gives an account of the general theorem. The two proofs which we present, while containing the same ingredients as that in [3], are, respectively, shorter and more direct. Section 3 contains a rather formal extension of the algebraic identity to an equality of Fredholm determinants for a simple class of functional integral operators; while not deep, it is a natural complement to the analytic determinant identity for differential operators.

2. Periodic solutions of difference equations

As in Section 1, K is an arbitrary field and $k \geqslant 1$ is a fixed natural number. Let V_i, $i \in \mathbb{Z}/k\mathbb{Z}$, be finite-dimensional vector spaces over K, indexed by the integers modulo k, and let $a_{i,j} : V_j \to V_i$, for $i,\ j \in \mathbb{Z}/k\mathbb{Z}$, be linear transformations. We think of (V_i) as a vector bundle V over the finite set $\mathbb{Z}/k\mathbb{Z}$ with space of sections the vector space $\Gamma(V) = \bigoplus_{i\in\mathbb{Z}/k\mathbb{Z}} V_i$. Thus a section is a k-tuple $x = (x_1, x_2, \ldots, x_k) \in V_1 \oplus V_2 \oplus \cdots \oplus V_k$. A linear operator $a : \Gamma(V) \to \Gamma(V)$ is defined by:

$$(ax)_i = \sum_j a_{i,j} x_j. \tag{2.1}$$

Suppose that this operator has order $\leqslant d$, $1 \leqslant d \leqslant k$, in the sense that $a_{i,i-r} = 0$ for $d < r \leqslant k$ and all i. We consider solutions of the difference equation

$$x = ax : x_i = a_{i,i-1}x_{i-1} + \cdots + a_{i,i-d}x_{i-d}. \tag{2.2}$$

The solutions are the fixed-points of a.

In the standard way we replace the dth order equation by a first order equation in d variables, defining a vector bundle W over $\mathbb{Z}/k\mathbb{Z}$ by

$$W_i = \bigoplus_{r=-d}^{-1} V_{i+1+r} = V_{i+1-d} \oplus \cdots \oplus V_i,$$

and a linear transformation $b_i : W_{i-1} \to W_i$ by

$$b_i(u_{i-d}, \ldots, u_{i-1}) = (v_{i+1-d}, \ldots, v_i),$$

where

$$v_{i+1+r} = \begin{cases} a_{i,i-d}u_{i-d} + \cdots + a_{i,i-1}u_{i-1} & \text{if } r = -1, \\ u_{i+1+r} & \text{if } -d \leqslant r < -1. \end{cases}$$

In these definitions, the integer index $r \in \mathbb{Z}$ is reduced (mod k) where appropriate. Solutions of (2.2) correspond to solutions of the first order difference equation:

$$y_i = b_i y_{i-1}, \qquad y \in \Gamma(W), \tag{2.3}$$

under the correspondence $y_i = (x_{i+1-d}, \ldots, x_i)$. To maintain the parallel with the system $a_{i,j}$, let us define $b_{i,j} : W_j \to W_i$ to be 0 if $j \neq i-1$, $b_{i,i-1} = b_i$, and let $b : \Gamma(W) \to \Gamma(W)$ be the associated endomorphism of the space of sections of W.

We think of (2.2) as a *delay equation* for x_i, $i \geqslant -d+1$, with initial data $x_{-d+1}, \ldots, x_0$. The initial data are propagated one step at a time by (2.3) to give the solution for all i.

Periodic solutions of (2.2) correspond to fixed-points of the monodromy (or the Poincaré) map:

$$m_0 = b_k \circ b_{k-1} \circ \cdots \circ b_1 : W_0 \to W_0.$$

Thus, clearly $\det(1-a) = 0$ if and only if $\det(1-m_0) = 0$. In fact, the determinants are equal.

Proposition 2.4 [3]. *In the situation described above*,

$$\det(1 - a : \Gamma(V) \to \Gamma(V)) = \det(1 - m_0 : W_0 \to W_0).$$

Examples 1.1 and 1.2 arise as the special cases in which $V_i = K^n$, $d = 1, 2$ respectively, $a_{i,i-1} = A_i$ and, for 1.2, $a_{i,i-2} = B_i$.

It is an easy exercise to establish the result for $d = 1$.

Proof of 2.4 in the case that $d = 1$. Now $1 - a$ maps $(x_1, \dots, x_k)$ to $(x_1 - a_1 x_k, \dots, x_k - a_k x_{k-1})$. Define an endomorphism f of $\Gamma(V)$ by $f(x_1, \dots, x_k) = (f_1 x_k, f_2 x_k, \dots, f_{k-1} x_k, 0)$, where $f_1 = a_1$ and $f_i = a_i f_{i-1}$.

Then $(1 - a)(1 + f)$ maps $(x_1, \dots, x_k)$ to $(x_1, x_2 - a_2 x_1, \dots, x_{k-1} - a_{k-1} x_{k-2}, (1 - m_0) x_k - a_k x_{k-1})$. The matrix is triangular, with determinant $\det(1 - m_0)$, and this determinant is equal to $\det(1 - a)$, since $\det(1 + f) = 1$. $\square$

Remark 2.5. If K has characteristic 0, the trace can be used to give another proof. For $\operatorname{tr}(a^r) = 0$ if r is not divisible by k, and $\operatorname{tr}(a^{sk}) = k \operatorname{tr}(m_0^s)$. It follows that

$$\begin{aligned} \det(1 - Ta) &= \exp\left(-\sum \frac{T^r}{r} \operatorname{tr}(a^r)\right) \\ &= \exp\left(-\sum \frac{T^{sk}}{s} \operatorname{tr}(m_0^s)\right) = \det(1 - T^k m_0) \end{aligned}$$

in the formal power series ring $K[[T]]$.

First proof of 2.4. It is clear that $\det(1 - m_0)$ remains unchanged if the bound on the order d is increased to k, since the monodromy map m_0 is zero on the extra summands. It suffices, therefore, to deal with the case $d = k$.

By taking V_i to be K^{n_i}, we can view the equality 2.4 as a polynomial identity, $F(X_{p,q}) = G(X_{p,q})$ say, in the matrix coefficients $X_{p,q}$, $1 \leqslant p, q \leqslant n = n_1 + \dots + n_k$, of a. There is no loss of generality in supposing that K is algebraically closed. The zero-sets of the polynomials $F(X_{p,q})$ and $G(X_{p,q})$ in $M_n(K)$ coincide, and $F(X_{p,q})$ $(= \det(I - [X_{p,q}]))$ is irreducible. Hence, by the Nullstellensatz, $G(X_{p,q})$ must be of the form $\lambda(F(X_{p,q}))^m$, for some non-zero $\lambda \in K$ and $m \geqslant 1$. By putting $X_{p,q} = 0$, one checks that $\lambda = 1$. Comparison of the coefficients of $X_{1,1}$ shows that $m = 1$. $\square$

Second proof of 2.4. From the special case $d = 1$, applied to b instead of a, we have:

$$\det(1 - b) = \det(1 - m_0).$$

The proof will be completed by verifying the following plausible equality.

Lemma 2.6

$$\det(1 - a : \Gamma(V) \to \Gamma(V)) = \det(1 - b : \Gamma(W) \to \Gamma(W)).$$

Proof. As in the first proof, we may restrict attention to the case where $d = k$. Let us set $W_i^{(r)} = V_{i-j} \subseteq W_i$, for $r = 0, \dots, k - 1$ and write $W^{(r)} = W_1^{(r)} \oplus \dots \oplus W_k^{(r)}$.

So each $W^{(r)}$ is equal to the direct sum $V_1 \oplus \cdots \oplus V_k$, and $\Gamma(W) = W^{(0)} \oplus \cdots \oplus W^{(k-1)}$. With respect to this decomposition b has the matrix form:

$$\begin{bmatrix} a^{(0)} & a^{(1)} & a^{(2)} & \cdots & a^{(k-1)} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

where $a^{(r)} : W^{(r)} \to W^{(0)}$ maps (x_{i-r}) to $(a_{i,i-r}x_{i-r})$. Thus $a^{(0)} + a^{(1)} + \cdots + a^{(k-1)} = a$ as an endomorphism of $V_1 \oplus \cdots \oplus V_k$.

The rest of the proof is an elementary matrix computation. Let g be the nilpotent endomorphism of $W^{(0)} \oplus \cdots \oplus W^{(k-1)}$ mapping $(x^{(0)}, \ldots, x^{(k-1)})$ to $((a^{(1)} + \cdots + a^{(k-1)})x^{(1)} + \cdots + (a^{(k-2)} + a^{(k-1)})x^{(k-2)} + a^{(k-1)}x^{(k-1)}, 0, \ldots, 0)$. Then $(1 + g)(1 - b)$ is lower triangular with diagonal entries $1 - a, 1, \ldots, 1$. $\square$

The same arguments show that

$$\det(1 - T^k m_0) = \det(1 - (Ta^{(1)} + T^2a^{(2)} + \cdots + T^k a^{(k)})) \in K[T].$$

Remark 2.7. Suppose that $a_{q,r} = 0$ for $1 \leqslant q \leqslant r < k$. Let $l_q : V_0 \to V_q$ be defined recursively, for $0 \leqslant q \leqslant k$, by:

$$l_0 = 1, \qquad l_q = a_{q,0}l_0 + a_{q,1}l_1 + \cdots + a_{q,q-1}l_{q-1} \quad \text{for } 1 \leqslant q \leqslant k.$$

Then $m_0 : W_0 \to W_0$ factors through V_0 and

$$\det(1 - m_0 : W_0 \to W_0) = \det(1 - l_k : V_0 \to V_0).$$

The restriction that the vector spaces V_i be finite-dimensional can be dropped. It suffices to assume that the linear maps $a_{i,j}$ are of finite rank.

Corollary 2.8. *Suppose that the vector spaces V_i are not necessarily finite-dimensional, but that each linear map $a_{i,j}$ has finite rank. Then the monodromy map m_0 is of finite rank and*

$$\det(1 - a : \Gamma(V) \to \Gamma(V)) = \det(1 - m_0 : W_0 \to W_0).$$

For one can split V as a direct sum $V' \oplus V''$ with V' finite-dimensional, such that $a_{i,j} : V'_j \oplus V''_j \to V'_i \oplus V''_i$ splits as $a'_{i,j} \oplus 0$.

Remark 2.9. The maps b_i will not, in general, be of finite rank. However, b^k will be of finite rank, and $\det(1 - b)$ still makes sense, as the determinant of $1 - b$ restricted to the image of b^r for any r such that b^r has finite rank. With this interpretation, 2.6 still holds.

3. Periodic solutions of delay equations

In this section we turn to analysis. Let the vector spaces V_i, for $i \in \mathbb{Z}/k\mathbb{Z}$, be separable complex Hilbert spaces. Suppose that the linear operators $a_{i,j} : V_j \to V_i$ are of trace class. Then a and the monodromy m_0 are also of trace class.

Proposition 3.1. *In the situation described in the text, we have an equality of Fredholm determinants*:

$$\det(1 - a) = \det(1 - m_0).$$

Proof. This follows either by approximation from the finite-dimensional theory or by reproducing the second proof of 2.4. □

Remark 3.2. If the operators $a_{i,j}$ are compact, but not necessarily of trace class, then we still have a topological theorem for $\mathbb{R}$-Hilbert spaces. Suppose that $1 - a$ and $1 - m_0$ have trivial kernel, so that zero is the only fixed-point of a. Then the Leray-Schauder fixed-point indices of a and m_0, defined as elements of $\{1, -1\} \subseteq \mathbb{Z}$ (and equal to the sign of the Fredholm determinant when the operators are of trace class), coincide, as can be seen by approximating the compact operators by operators of finite rank in the norm topology. See, for example, [5].

We shall apply this result to a particularly simple class of integral equations. Let δ be a real number in the range: $0 < \delta < 1$. Suppose that $\kappa : \mathbb{R} \times (-\infty, 0] \to \mathbb{C}$ is a continuous function with support in $\mathbb{R} \times (-\delta, 0)$ such that $\kappa(s + 1, t) = \kappa(s, t)$ for all s, t. Consider the delay equation:

$$x(s) = \int_{-\delta}^{s} \kappa(s, t - s)x(t)\,\mathrm{d}t \quad \text{for } s > 0, \tag{3.3}$$

where $x : [-\delta, \infty) \to \mathbb{C}$ is continuous on $(0, \infty)$ and of class L^1 on $[-\delta, 0]$. A routine application of the contraction mapping theorem shows that, given $y \in L^1[-\delta, 0]$, there is a unique continuous function $x : [0, \infty) \to \mathbb{C}$ such that

$$x(s) = \int_{0}^{s} \kappa(s, t - s)x(t)\,\mathrm{d}t + \int_{-\delta}^{0} \kappa(s, t - s)y(t)\,\mathrm{d}t. \tag{3.4}$$

More generally, there is a unique continuous function $\mu : [0, \infty) \times [-\delta, 0] \to \mathbb{C}$ such that

$$\mu(s, t) = \int_{0}^{s} \kappa(s, u - s)\mu(u, t)\,\mathrm{d}u + \kappa(s, t - s),$$

and the solution of (3.4) is given by

$$x(s) = \int_{-\delta}^{0} \mu(s, t)y(t)\,\mathrm{d}t.$$

Thus far, we have used the continuity of κ, but not the condition: $\kappa(s, 0) = 0$.

We now study periodic solutions of the delay equation (3.3) with period 1. Let $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be the doubly-periodic continuous function such that $\alpha(s, t) = \kappa(s, t - s)$ for $s - 1 \leqslant t \leqslant s$ and $\alpha(s, t + 1) = \alpha(s, t)$ for all $s,\ t$. (Continuity depends on the vanishing of $\kappa(s, 0)$.) Then the solutions of period 1 may be described as fixed-points of the operator $a : L^2(\mathbb{R}/\mathbb{Z}) \to L^2(\mathbb{R}/\mathbb{Z})$ given by

$$(ax)(s) = \int_0^1 \alpha(s, t)x(t)\,\mathrm{d}t$$

or in terms of the monodromy map $m_0 : L^2[-\delta, 0] \to L^2[-\delta, 0]$ given by

$$(m_0 y)(s) = \int_{-\delta}^0 \mu(s + 1, t)y(t)\,\mathrm{d}t.$$

Both a and m_0 are of trace class, so that the Fredholm determinants of $1 - a$ and $1 - m_0$ are defined. Moreover, $\det(1 - a)$ and $\det(1 - m_0)$ depend continuously on the kernel κ (with the supremum norm). (This is clear for a; for m_0, the continuous dependence of μ on κ is again part of the contraction mapping theory.)

Proposition 3.5. *There is an equality of Fredholm determinants*:

$$\det(1 - m_0 : L^2[-\delta, 0] \to L^2[-\delta, 0]) = \det(1 - a : L^2(\mathbb{R}/\mathbb{Z}) \to L^2(\mathbb{R}/\mathbb{Z})).$$

Proof. We have made the assumption that $\kappa(s, 0) = 0$. By continuity, it is enough to prove the result for a kernel κ such that there is an integer $k > 1/\delta$ with $\kappa(s, t) = 0$ for $-1/k \leqslant t \leqslant 0$. In this case the monodromy m_0 can be written down explicitly as an iterated integral.

Let V_i be the Hilbert space $L^2[(i - 1)/k, i/k]$, for $1 \leqslant i \leqslant k$, and define $a_{i,j}$ by

$$(a_{i,j}x)(s) = \int_{(j-1)/k}^{j/k} \alpha(s, t)x(t)\,\mathrm{d}t.$$

Note that $a_{i,i} = 0$. We apply 3.1 with $d = k$. The direct sum $V_1 \cdots \oplus V_k$ is identified with $L^2(\mathbb{R}/\mathbb{Z}) = L^2[0, 1]$ and W_0 with $L^2[-1, 0]$. The determinant of $1 - m_0$ on $L^2[-1, 0]$ is the same as that on $L^2[-\delta, 0]$, because m_0 is zero on the first factor of $L^2[-1, -\delta] \oplus L^2[-\delta, 0] = L^2[-1, 0]$. $\square$

Remark 3.6. In Atiyah's result [1] the finite-dimensional determinant of the holonomy is much simpler than the regularized ζ-determinant of the Dirac operator. Here, in contrast, although both determinants in 3.5 are infinite-dimensional Fredholm determinants, the determinant of the operator $1 - a$ is the more tractable, being given explicitly by the kernel α. We have the integral formula due to Fredholm (see, for example, [10]):

$$\det(1 - a) = \sum_{r \geqslant 0} (-1)^r \operatorname{tr} \Lambda^r a,$$

where

$$\operatorname{tr} \Lambda^r a = \int_{0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_r \leqslant 1} \det[\alpha(t_i, t_j)]_{1 \leqslant i, j \leqslant r} \, \mathrm{d}t_1 \, \mathrm{d}t_2 \, \ldots \, \mathrm{d}t_r .$$

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